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Towards inflation and dark energy cosmologies from modified Gauss-Bonnet theory

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ABSTRACT: We consider a physically viable cosmological model that has a field dependent Gauss-Bonnet coupling in its effective action, in addition to a standard scalar field potential. The presence of such terms in the four dimensional effective action gives rise to several novel effects, such as a four dimensional flat Friedmann-Robertson-Walker universe undergoing a cosmic inflation at the early epoch, as well as a cosmic acceleration at late times. The model predicts, during inflation, spectra of both density perturbations and gravitational waves that may fall well within the experimental bounds. Furthermore, this model provides a mechanism for reheating of the early universe, which is similar to a model with some friction terms added to the equation of motion of the scalar field, which can imitate energy transfer from the scalar field to matter.

KEYWORDS: Inflation, Dark Energy Cosmologies, Classical Theories of Gravity

Contents

1. Introduction

Although Einstein's theory has been proven to be remarkably simple and successful as a classical theory of gravitational interactions, there are several observational facts which it has failed to elucidate. These cosmological conundrums include both cosmic inflation, or a period of accelerated expansion in the early universe, and a recent acceleration in the expansion of the universe.

Inflation in the early universe is a very attractive proposal for explaining the present large scale homogeneity and high degree of isotropy of the universe (one part in $100,000$), in addition to the observed spectrum of density perturbations, which is usually attributed to a scalar field rolling down a shallow potential. Similarly, the current acceleration of the universe, as indicated by recent cosmological results [1], is usually attributed to some form of cosmic fluid having a large and smoothly distributed negative pressure, usually called *dark energy* or *dark pressure*.

Cosmologists have long wondered why/how the universe has been recently accelerating: is it due to a pure cosmological constant term, or due to some sort of negative pressure generated by one or more dynamical scalar fields, or something else? In recent years, different explanations have been provided for both inflation and the current epoch of acceleration: some examples of recent interest include brane-world modification of Einstein's general relativity (GR), including a 5d DGP (Dvali-Gabadadze-Porrati) model [2]. The names of the dark energy candidates run the gamut from $f(R)$ gravity [3] (modifying in a very radical manner the Einstein's GR itself) to ghost condensates (the idea which ncludes a more or less disguised non-locality) [4]. Many of these new proposals are pathological and do not appear more appealing than the two long envisioned alternative models of dark energy: a cosmological constant [5] and a slowly varying Λ-term [6].

The cosmological constant is a pure dark energy or vacuum energy, while the variable Λ-term is some kind of exotic matter or a slowly varying potential of a scalar field, usually referred to as *quintessence* [7]. This last category can comprise a Casimir energy or vacuum polarization effect from additional compact or curved non-compact spatial dimensions, which only weakly couple to ordinary matter *in contrast* to most tentative quintessence models, including k-essence [8] or curvature quintessence. In a brane world scenario [9], for instance, the vacuum energy (or the dark energy) may be viewed as a smooth *brane-tension* if our universe is a 3-brane embedded in higher dimensional spacetimes.

Indeed, the past decade has witnessed significant progress in the building of inflationary models as extensions of standard cosmology, to accommodate the effects of dark energy. However, most of the inflationary type potentials studied in the literature are picked up in a very *ad hoc* fashion, rather than constructing such a potential as a valid solution of the field equations that follows, for instance, from low energy string effective actions. In this paper we initiate work in this direction.

2. Action and equations of motion

For a given scalar field σ with a self-interaction potential $V(\sigma)$, an effective action as the low energy approximation of a fundamental theory of gravity and fields is written as

$$S = \int d^4x \left[\sqrt{-g} \left(\frac{1}{2\kappa^2} R - \frac{1}{2} (\nabla \sigma)^2 - V(\sigma) \right) \right], \tag{2.1}$$

where κ is the inverse Planck scale $M_P^{-1} = (8\pi G_N)^{1/2}$ and σ is a classical scalar field whose stress-energy tensor acts like a time-varying Λ. When studying the dynamics of an inflationary universe, the choice of the field potential $V(\sigma)$ is of particular interest. However, the origin and exact nature of the field σ and the functional form of $V(\sigma)$ that acts as an extra source of gravitational repulsion (or dark energy) are not precisely known. A real motivation for the gravitational action of the form (2.1) arises from the following interesting observation. Inflation with the dynamics of a scalar field with a self-interaction potential, $V(\sigma)$, provides a negative pressure to drive the accelerating expansion of the universe [10] as well as a mechanism for generating the observed density perturbations [11].

The effective action (2.1) is found to be remarkably simple, but it excludes, at least, one important piece, which is the coupling between the scalar field σ and the Riemann curvature tensor. This is because the field σ, if its vacuum expectation value is to describe the size and shape of the internal compactification manifold, generically couples with the curvature squared terms [12,13]. To this end, one welcomes the idea that a dark energy and its associated cosmic acceleration is due to a modification of general relativity such that the scalar field σ couples to gravity via the curvature squared terms in the Gauss-Bonnet (GB) combination. The effective action for the system may be taken to be

$$S = \int d^4x \left[\sqrt{-g} \left(\frac{1}{2\kappa^2} R - \frac{\gamma}{2} (\nabla\sigma)^2 - V(\sigma) + (\lambda - \xi(\sigma)) \mathcal{G} + \cdots \right) \right], \qquad (2.2)$$

where $\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is the Gauss-Bonnet invariant and $V(\sigma)$ and $\xi(\sigma)$ are general functions of σ. Dots represent other possible contribution to the gravitational action, such as $\zeta(\sigma)(\nabla\sigma)^4$ [13,14], which we drop here for simplicity. This is justified, since the quartic term $(\nabla\sigma)^4$ usually decays faster than the GB curvature invariant. The most desirable feature of GB type curvature corrections is that only the terms which are the second derivatives of the metric (or their product) appear in the field equations: a feature perhaps most important in order to make a theory of scalar-tensor gravity ghost free.

For some, a weakness of this approach to model dark energy may reside in the standard argumentation for this particular kind of modification of Einstein's theory, especially when one remembers that string/M theory predicts not only the fourth derivative gravitational term, like a GB invariant, but also higher-order terms. As is known, the field dependence of the coupling $\xi(\sigma)$ has its origin in the variation of the background spacetime, and, in a spatially flat spacetime background, the GB term is subject to a non-renormalization theorem which implies that all moduli dependent higher loop contributions (e.g. terms cubic and higher order in Riemann tensor) vanish [12,15].

We anticipate that the Gauss-Bonnet invariant $\mathcal{G}$ decays faster than the coupling $\xi(\sigma)$ grows, so that the term $\xi(\sigma)\mathcal{G}$ is only subdominant in the effective action. Moreover, the Gauss-Bonnet term is a topological invariant in four dimensions if $\xi(\sigma)$ is a constant, but not if $\xi(\sigma)$ is a dynamical variable.

In an influential paper, Antoniadis et al. [15] demonstrated the existence of cosmological solutions which avoid the initial singularity and are consistent with the perturbative treatments of the string effective actions; see, e.g.; [16] for other generalizations. The essential ingredient of their method is a field-dependent Gauss-Bonnet coupling, $\xi(\sigma)$, where σ characterizes the overall size and the shape of the internal compactification manifold. In the present paper we go one step further by demonstrating with new exact non-singular solutions that inflation in the early universe, as well as a cosmic acceleration at late times, may be explained by introducing a scalar potential $V(\sigma)$ for the modulus field σ, in addition to a field dependent Gauss-Bonnet coupling, $\xi(\sigma)$. We have reported some of the key results of our work in ref. [17]. The present paper will further elaborate the details of the model. For mathematical simplicity, we henceforth define the function $f(\sigma) \equiv \lambda - \xi(\sigma)$.

While the field potential, $V(\sigma)$, was absent in some original string amplitude computations, e.g. [12,13], implying that $V(\sigma)$ is a phenomenologically motivated field potential, it is quite possible that, in the presence of additional sources (like branes, fluxes), the string effective action would incorporate a non-trivial field potential, $V(\sigma)$, as is the case revealed recently from string theory cosmic landscape [18, 19]. Unfortunately, we do not have a precise knowledge about the field potential $V(\sigma)$. In string theory context, any such potential might take into account some non-perturbative effects, as arising from the effects of branes/fluxes present in the extra dimensions. In this sense, one may consider our model as *string-inspired*.

An interesting question that we would like to ask is what new features would a dynamical Gauss-Bonnet coupling introduce, and how can it influence cosmological evolution? Recently, it was suggested in [20] that the action (2.2) with some specific choices of the potential and the scalar-GB coupling, namely,

$$V(\sigma) = V_0 e^{-\sigma(t)/\sigma_0}, \quad f(\sigma) = f_0 e^{\sigma(t)/\sigma_0}, \tag{2.3}$$

may be used, in four dimensions, to explain the current acceleration of the universe, including the phantom crossing the phantom divide ($w = -1$), with effective (cosmological constant or quintessence) equation of state of our universe; also see [21] for other interesting generalizations.

In this paper, instead of choosing particular functional forms for $V(\sigma)$ and $f(\sigma)$, as above, we present exact cosmological solutions that respect the symmetry of the field equations which follow from (2.2). The graviton equation of motion derived from the action (2.2) may be expressed in the following form (see, for example, ref. [22])

$$
\begin{aligned}
0 = {} & \frac{1}{2\kappa^2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) - \frac{\gamma}{2}\left(\nabla_\mu\sigma\nabla_\nu\sigma - \frac{1}{2}g_{\mu\nu}(\nabla\sigma)^2\right) + \frac{1}{2}f(\sigma)(4X_{\mu\nu} - g_{\mu\nu}\mathcal{G}) \\
& + \frac{1}{2}g_{\mu\nu}V(\sigma) + 2(g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu)(f(\sigma)R) - 4g_{\mu\nu}\nabla^\lambda\nabla^\rho(f(\sigma)R_{\lambda\rho}) - 4\nabla^2(f(\sigma)R_{\mu\nu}) \\
& + 4\nabla^\rho\nabla_\mu(f(\sigma)R_{\nu\rho}) + 4\nabla^\rho\nabla_\nu(f(\sigma)R_{\mu\rho}) + 4\nabla^{(\rho}\nabla^{\lambda)}(f(\sigma)R_{\mu\rho\nu\lambda}),
\end{aligned}
\tag{2.4}
$$

where $X_{\mu\nu} \equiv RR_{\mu\nu} + R_{\mu\rho\sigma\lambda}R_\nu{}^{\rho\sigma\lambda} - 4R_\mu{}^\rho R_{\nu\rho}$. The equation of motion for the scalar field σ is similarly given by

$$0 = \gamma\nabla^2\sigma - \frac{dV(\sigma)}{d\sigma} + \frac{df(\sigma)}{d\sigma}\mathcal{G}. \tag{2.5}$$

Next, we consider a four-dimensional background spacetime defined by the standard Friedmann-Robertson-Walker metric:

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3}(dx^i)^2. \tag{2.6}$$

In this background,

$$\mathcal{G} = 24H^2(\dot{H} + H^2), \tag{2.7}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and $\dot{a} \equiv \frac{da}{dt}$. The quantity $4X_{\mu\nu} - g_{\mu\nu}\mathcal{G}$ vanishes. The $(\mu,\nu) = (t,t)$ and (x,x) components of the field equations have the following forms

$$0 = -\frac{3}{\kappa^2}H^2 + \frac{\gamma}{2}\dot{\sigma}^2 + V(\sigma) - 24\dot{\sigma}H^3\frac{df}{d\sigma}, \tag{2.8}$$

$$0 = \frac{1}{\kappa^2}\left(2\dot{H} + 3H^2\right) + 8H^2\left(\ddot{\sigma}\frac{df}{d\sigma} + \dot{\sigma}^2\frac{d^2f}{d\sigma^2}\right) + 16H\dot{\sigma}\frac{df}{d\sigma}(\dot{H} + H^2) + \frac{\gamma}{2}\dot{\sigma}^2 - V(\sigma). \tag{2.9}$$

The time evolution equation for $\sigma(t)$ (cf equation (2.5)) can be written as

$$\begin{aligned}
0 &= -\gamma\left(\ddot{\sigma} + 3H\dot{\sigma}\right)\dot{\sigma} - \frac{\mathrm{d}V(\sigma)}{\mathrm{d}t} + \frac{\mathrm{d}f(\sigma)}{\mathrm{d}t}\mathcal{G} \\
\Rightarrow\quad & \gamma\left(\dot{\sigma}\ddot{\sigma} + 3H\dot{\sigma}^2\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(V(\sigma) - f(\sigma)\mathcal{G}\right) + f(\sigma)\frac{\mathrm{d}\mathcal{G}}{\mathrm{d}t} = 0 \\
\Rightarrow\quad & \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\gamma}{2}\dot{\sigma}^2 + \Lambda(\sigma)\right) + 6H\left(\frac{\gamma}{2}\dot{\sigma}^2\right) + \delta = 0,
\end{aligned} \tag{2.10}$$

where we have defined

$$\Lambda(\sigma) \equiv V(\sigma) - f(\sigma)\mathcal{G}, \quad \delta \equiv f(\sigma)\frac{\mathrm{d}\mathcal{G}}{\mathrm{d}t}. \tag{2.11}$$

We will call $\Lambda(\sigma)$ an effective potential. Due to the Bianchi identity, one of the field equations (2.8)-(2.10) is redundant and hence may be discarded without loss of generality. In the limit $f(\sigma)H^2 \to 0$, the action (2.2) reduces to (2.1).

The δ term in (2.10) may account for the creation of particles due to time variation of $\mathcal{G}$. A friction-like term like this was first introduced in [23] on phenomenological grounds and a question was subsequently raised in [24] about the physical origin of such term; in our model, the δ term is a clear manifestation of a non-trivial coupling (or back-reaction) between the field σ and the time-varying Gauss-Bonnet curvature invariant. This term represents a clear advantage of (2.2) over (2.1) as a cosmological action.

3. Scalar field as a perfect fluid

In a spatially flat background, the Einstein tensor, $G_{\mu\nu}$, has components $G_{00} = 3H^2$, $G_{ii} = -a^2(2\dot{H} + 3H^2)$. Assuming that the stress-energy tensor $T_{\mu\nu}$ is described by a perfect fluid of the form $T_{00} = \rho, T_{ii} = a^2 p$, we find

$$\rho = \frac{3H^2}{\kappa^2}, \qquad p = -\frac{2\dot{H} + 3H^2}{\kappa^2}. \tag{3.1}$$

We also immediately find from equations (2.8)-(2.9)

$$\rho = \frac{\gamma}{2}\dot{\sigma}^2 + V(\sigma) - 24H^3\dot{f}, \tag{3.2}$$

$$p = \frac{\gamma}{2}\dot{\sigma}^2 - V(\sigma) + 8\frac{\mathrm{d}}{\mathrm{d}t}(H^2\dot{f}) + 16H^3\dot{f}, \tag{3.3}$$

where $f = f(\sigma)$. Note that this differs from the usual expression, due to the field-dependent coupling constant, $f(\sigma)$. To guarantee that the energy density of the scalar field is always positive, we require $\dot{f}H < 0$.

We assume the scalar field obeys an equation of state, with an equation of state parameter given by

$$w \equiv \frac{p}{\rho} = -1 - \frac{2}{3}h = \frac{2q - 1}{3}, \tag{3.4}$$

where $q \equiv -a\ddot{a}/\dot{a}^2$ is the deceleration parameter. A minimal criterion for getting an accelerating expansion is $\rho + 3p < 0$, which is clearly a violation of the strong energy condition for some time-like vectors ξ^μ; the latter states that $T_{\mu\nu}\xi^\mu\xi^\nu \geq \frac{1}{2}T_\lambda{}^\lambda\xi^\mu\xi^\nu$ or equivalently $\rho + 3p \geq 0$ and $\rho + p \geq 0$. Note that

$$\rho + 3p = 2\gamma\dot{\sigma}^2 - 2V(\sigma) + 24\frac{\mathrm{d}}{\mathrm{d}t}(NH^2\dot{f}), \tag{3.5}$$

where $N = \int H\,\mathrm{d}t$, so that $\mathrm{d}N/\mathrm{d}t = H$. For $\dot{f} = 0$, the acceleration condition $\rho + 3p < 0$ holds when $V(\sigma) > \gamma\dot{\sigma}^2$. In the case $\dot{f} \neq 0$, however, whether the condition $V(\sigma) > \gamma\dot{\sigma}^2$ is sufficient or not depends on the sign of the time-derivative of the coupling, $f(\sigma)$. For a canonical scalar field (i.e. $\gamma > 0$), both $\dot{f} < 0$ and $\dot{H} \leq 0$ hold, in general. The acceleration condition $\rho + 3p < 0$ is satisfied for $\dot{f}(\sigma) < 0$. Similarly, we find,

$$\rho + p = \gamma\dot{\sigma}^2 + 8\frac{\mathrm{d}}{\mathrm{d}t}\left(H^2\dot{f}\right) - 8H^3\dot{f} > 0, \tag{3.6}$$

and not zero as it would be in a spacetime which is exactly de Sitter: $\gamma = 0$, $f(\sigma) = \text{const.}$ The null energy condition $T_{\mu\nu}\chi^\mu\chi^\nu \geq 0$ (for some null vectors χ^μ), or equivalently $p + \rho \geq 0$ may be violated by allowing $\gamma < 0$, or instead by taking $\ln(H^2\dot{f}) + \int\gamma\dot{\sigma}^2\mathrm{d}t < N$.

4. General solutions

To simplify the study of the model, we define the following dimensionless variables:

$$x = \frac{\gamma\kappa^2}{2}\left(\frac{\dot{\sigma}}{H}\right)^2, \quad y = \frac{\kappa^2 V(\sigma)}{H^2}, \quad u = 8\kappa^2 f(\sigma)H^2, \quad h = \frac{\dot{H}}{H^2}. \tag{4.1}$$

The equations of motion, (2.8)-(2.10), then form a set of second order differential equations (see the Appendix). In the absence of the GB coupling, so $u = 0$, we find a simple relationship:

$$y = 3 + h, \qquad x = -h. \tag{4.2}$$

A physically more intriguing case is $u \neq 0$, for which the variables satisfy

$$y = 3 + h + \frac{1}{2}\left[u'' + (5 - h)u' - 2(h^2 + h' + 5h)u\right], \tag{4.3}$$

$$x = -h - \frac{1}{2}\left[u'' - (1 + h)u' - 2(h^2 + h' - h)u\right], \tag{4.4}$$

where the prime denotes a derivative w.r.t logarithmic time or number of e-folds $N \equiv \ln(a(t)/a_0)$. Thus there can exist a large class of solutions with different $u(N)$.

The quantity $\Lambda(\sigma) \equiv V(\sigma) - f(\sigma)\mathcal{G}$ may act as an effective potential for the field σ. It is clear that $\Lambda(\sigma)$ is a second order differential equation with non-constant coefficients:

$$\kappa^2 \Lambda(\sigma) = H^2 \left(y - 3u(1 + h)\right)$$
$$= \frac{H^2}{2}\left[u'' + (5 - h)u' - 2(8h + h^2 + h' + 3)u\right] + H^2(3 + h). \tag{4.5}$$

When solving (4.5) for u, one finds that the *homogeneous* part of the solution corresponds to solving the *complementary* differential equation,

$$u'' + (5 - h)u' - 2(8h + h^2 + h' + 3)u = 0. \tag{4.6}$$

The *homogeneous* solution is the full solution when one makes the ansatz

$$\kappa^2 \Lambda(\sigma) = H^2 (3 + h), \tag{4.7}$$

which removes all *nonhomogeneous* terms. Of course, $f(\sigma) = 0$ (i.e. $u = 0$) is the trivial solution for (4.6), which corresponds to the absence of Gauss-Bonnet coupling. Note that by setting $\kappa^2 \Lambda = H^2 (3 + h)$ one effectively makes the ansatz, $V(\sigma) = f(\sigma)\mathcal{G} + M_P^2 H^2(\sigma)(3 + h)$, reducing the number of arbitrary parameters of the model to one; fixing h alone will fix the function $u(\sigma(N))$, or vice versa. A salient feature of our construction of cosmological solutions is that, while the contributions coming from both the field potential $V(\sigma)$, and the GB potential term, $V_{GB}(\sigma)$, may be large separately, the effective potential, $\Lambda(\sigma)$, can be exponentially close to zero at late times, as it dynamically relaxes to a small value after a sufficiently large number of e-folds of expansion.

A commonly discussed alternative is the following. One solves the (modified) Einstein equations by making assumptions about the (functional) form of the field potential $V(\sigma)$ as well as the GB coupling $f(\sigma)$, which may be motivated by the leading order terms obtained (see, for example [20, 25]), by a time-dependent (cosmological) compactification of classical supergravities. However, in this fashion, one may find a good approximation at each energy scale, but the corresponding solution will have little relevance when one attempts to study a wider range of scales, including an inflationary era.

4.1 Homogenous solution with constant h

As a first reasonable approximation at low energy, we make the ansatz $h \equiv \dot{H}/H^2 = H'/H \simeq \mathrm{const} < 0$. More specifically, when $h = h_0$, the Hubble parameter is given by

$$H = \mathrm{e}^{\int h \, \mathrm{d}N} = H_0 \, \mathrm{e}^{h_0 N} \equiv (c_0 - h_0 t)^{-1} , \qquad (4.8)$$

where H_0 (or c_0) is an integration constant. Clearly, for $t > 0$, we require $h_0 < 0$ (and also $H_0 > 0$). The scale factor is given by $a(t) = a_0(c_0 - h_0 t)^{-1/h_0}$, implying that the universe accelerates when $-1 < h_0 < 0$. For $t \simeq 0$ (and/or $h_0 \simeq 0$), the size of the universe in this regime grows approximately as e^{Ht}, where $H \sim H_0 \sim c_0^{-1}$. Imposing (4.6), we find the homogenous solution:

$$u = u_1 \, \mathrm{e}^{c_1 N} + u_2 \, \mathrm{e}^{c_2 N} \qquad (4.9)$$

$$y = 3 + h_0 + 3(1 + h_0) \left(u_1 \, \mathrm{e}^{c_1 N} + u_2 \, \mathrm{e}^{c_2 N} \right) , \qquad (4.10)$$

$$x = -\frac{3}{2} \left(7 + 5h_0 + \sqrt{9h_0^2 + 54h_0 + 49} \right) u_1 \, \mathrm{e}^{c_1 N}$$
$$- \frac{3}{2} \left(7 + 5h_0 - \sqrt{9h_0^2 + 54h_0 + 49} \right) u_2 \, \mathrm{e}^{c_2 N} - h_0 , \qquad (4.11)$$

where

$$c_1 = \frac{1}{2} \left(h_0 - 5 + \sqrt{9h_0^2 + 54h_0 + 49} \right) ,$$
$$c_2 = \frac{1}{2} \left(h_0 - 5 - \sqrt{9h_0^2 + 54h_0 + 49} \right) , \qquad (4.12)$$

and u_1, u_2 are arbitrary at this stage. Whether the coupling function u is decreasing or increasing with N depends on the value of h_0. In particular, for $h_0 \sim 0$, we have $c_1 \sim -6$ and $c_2 \sim 1$, in which case the second term on the rhs of (4.9) increases with N. To rescue it, one might require to set $u_2 \sim 0$. However, for $h_0 \sim -1$ (and hence $w \sim -1/3$), since $c_1 \to -2$ and $c_2 \to -4$, only the first term on rhs. of (4.9) is relevant for large N, unless $u_2 \gg u_1$. For purely exponential solutions we require that u_1 and u_2 be real and $9h_0^2 + 54h_0 + 49 \geq 0$. This means that either

$$h_0 \leq -3 - \frac{4\sqrt{2}}{3} \simeq -4.88 \quad \mathrm{or} \quad h_0 \geq -3 + \frac{4\sqrt{2}}{3} \simeq -1.11, \qquad (4.13)$$

implying that $w > 2.2$ or $w < -0.26$. However, only for $h_0 > -1$ (and hence $w < -1/3$), the universe is accelerating.

On the other hand, if $9h_0^2 + 54h_0 + 49 < 0$, then the solution is written as

$$u(N) = e^{(h_0 - 5)N/2} \left[k_1 \sin \left(N \sqrt{-(9h_0^2 + 54h_0 + 49)} \right) \right.$$
$$\left. + k_2 \cos \left(N \sqrt{-(9h_0^2 + 54h_0 + 49)} \right) \right], \qquad (4.14)$$

where k_1, k_2 are real. We see that in the latter case for $h_0 < 5$ the GB contribution at late times (large N) becomes vanishingly small. This is automatically satisfied as

$$-4.88 < h_0 < -1.11, \tag{4.15}$$

implying that $-0.26 < w < 2.2$. Equations (4.9)-(4.12) provide, in some sense, a natural solution for Einstein's equations, as the effective cosmological constant, $\Lambda(\sigma)$, is dynamical and becomes arbitrarily small with time, while simultaneously they provide the simplest forms for the GB scalar coupling, $f(\sigma)$, the field potential, $V(\sigma)$, and the kinetic energy, $K(\sigma)$. However, this scenario cannot hope to effectively describe more than one epoch in the history of the universe, as h is not dynamical.

4.2 Homogenous solution with dynamical h

Motivated by the form of the previous solution we now make no assumptions about the form of the (slow-roll type) variable h, but instead consider the ansatz that during a given epoch we may make the approximation:

$$u(\sigma) \equiv f(\sigma) H^2 = (\lambda - \xi(\sigma))H^2 \approx u_t\, \mathrm{e}^{\alpha_t N}, \tag{4.16}$$

where the *time*, t, can be *early* or *late*, where $|u_{early}| \gg |u_{late}|$ and $|\alpha_{early}| > |\alpha_{late}|$. The Gauss-Bonnet invariant, $\mathcal{G} = 24H^2(H^2 + \dot{H}) = 24\left(\frac{\dot{a}}{a}\right)^2 \frac{\ddot{a}}{a}$, which is positive for the accelerating solutions (i.e. $\ddot{a} > 0$), decays faster than the coupling $\xi(\sigma)$ blows up, but it may well be that H^2 decays slower than $f(\sigma)$ (or $\xi(\sigma)$). In the case $f(\sigma) \sim 1/H^2$, $\alpha \simeq 0$.

One may motivate the above ansatz for the coupling $u(\sigma)$ in two different ways. Firstly, the function $f(\sigma)$ as a solution to a second order differential equation in $u(N)$ can have two different branches, which may dominate at different time scales, as can be seen above in the case $h \equiv \dot{H}/H^2 \simeq$ const. Secondly, in a known form for the coupling $f(\sigma)$ ($\equiv \lambda - \xi(\sigma)$), derived from heterotic string theory, the function $\xi(\sigma)$ may be given by [15, 16]

$$\xi(\sigma) = \delta \ln[2\mathrm{e}^\sigma \eta^4(i\mathrm{e}^\sigma)], \tag{4.17}$$

where δ is a constant and $\eta(b)$, with $b \equiv i\mathrm{e}^\sigma$ is the Dedekind η-function which is defined by $\eta(b) = \mathrm{e}^{i\pi b/12} \prod_{n=1}^\infty \left(1 - \mathrm{e}^{2i\pi nb}\right)$. As can be seen from the plots in figure 1, $\xi(\sigma)/\delta$ can be well approximated by $\ln 2 - \frac{2\pi}{3}\cosh(\sigma)$, while $(d\xi/d\sigma)/\delta$ by $-\frac{2\pi}{3}\sinh|\sigma|$.

In the expansion of the function $\xi(\sigma)$ (and hence $f(\sigma)$), the term proportional to $\mathrm{e}^{-\sigma}$ can dominate at early times, $\sigma \ll 0$, while the term proportional to e^σ can dominate at late times, $\sigma \gg 0$. The fact that such an expansion is typical of string effective actions represents, in our opinion, an interesting aspect of such models and implies the existence of two periods of accelerating expansion of the universe. Though the function $\xi(\sigma)$ is symmetric about $\sigma \to -\sigma$, the coupling $u(\sigma)$ is not, since the Hubble parameter H is a monotonically decreasing function of N ($\equiv \ln(a)$), or the field σ, and $H_{early} \sim 10^{23}\mathrm{eV} \gg H_{late} \sim 10^{-33}\mathrm{eV}$.
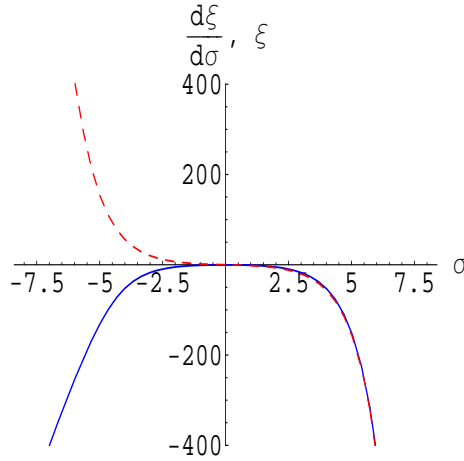
Figure 1: The function $\xi(\sigma)$ (solid line) is symmetric about $\sigma \to -\sigma$, while its first derivative, $d\xi/d\sigma$ (dotted line) is antisymmetric. Note that in these plots we have set $\delta = 1$.

By solving equation (4.7), or equivalently equation (4.6), we find

$$h(N) = -\hat{\beta} + \beta \tanh \beta \Delta N, \qquad (4.18)$$

where $\Delta N \equiv N - N_t$, with N_t being an integration constant, analogous to $\ln(a_0)$ in (A.5), and

$$\hat{\beta} = \frac{16 + \alpha}{4}, \quad \beta = \frac{\sqrt{9\alpha^2 + 72\alpha + 208}}{4}. \qquad (4.19)$$

Here we have suppressed the subscripts referring to the time on α. Because of our ansatz for $u(\sigma(N))$ of the form (4.16) the solution (4.18) is a good approximation at *early* or *late* epochs only. Of course, one could make a more complicated ansatz for the function $u(\sigma)$, motivated by some specific particle physics models, and find the corresponding solution for $h(N)$. We believe that the solution we have found above is sufficiently simple to explain the evolution of the universe both at early time $N \gtrsim N_{early}$ and at late times $N \gtrsim N_{late}$.

From (4.18), we easily find that

$$H(N) = e^{\int h(N) dN} = H_0 \, e^{-\hat{\beta} N} \cosh \beta \Delta N, \qquad (4.20)$$

where, again, the time t can be *early* or *late*. Writing the expressions for $y(N)$ and $x(N)$ is straightforward, using eqs. (4.3)-(4.4). Analogous to an inflationary type solution induced by a conformal-anomaly [26], the solutions given above are singularity-free. The effective potential therefore takes the form

$$\Lambda(\sigma(N)) = \frac{H_0^2}{\kappa^2} \left[3 - \hat{\beta} + \beta \tanh \beta \Delta N \right] (\cosh \beta \Delta N)^2 \, e^{-2\hat{\beta} N}. \qquad (4.21)$$

This possesses a local maximum around $0.15 \lesssim \Delta N \lesssim 1$, depending upon the value of the slope parameter, α (see figure 2) and generically decreases to zero with increasing N. In

fact, $\Lambda(\sigma)$ is almost flat for $\Delta N > 1$. As we are assuming that $N_{early} \ll N_{late}$, the local maxima and 'global minima' of the effective potential can be at different energy scales for the early-time and late-time universes. The kinetic energy, $K(\sigma)$, decreases with ΔN for $6 < \alpha < 0$, and increases for $0 < \alpha < 1$. The dimensionless variable, $x = K/H^2$, is approximately constant with $\Delta N \gtrsim 10$ for a wide range of α.
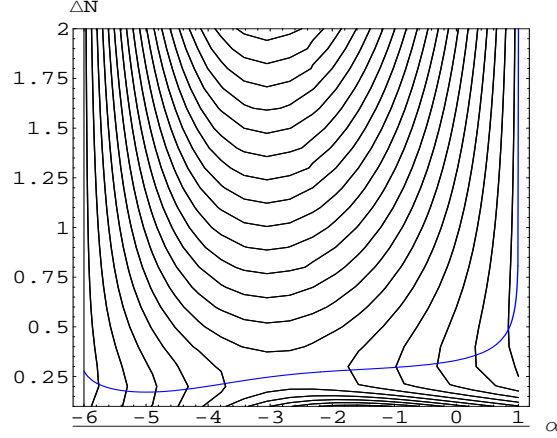

Figure 2: The contour plots of the effective potential $\Lambda(\sigma)/H_0^2$ with the height in log scale. The single solid (blue) line denotes where $d\Lambda/dN = 0$, giving the local maximum of the potential w.r.t. ΔN. For larger ΔN, the potential generically decreases exponentially to zero.
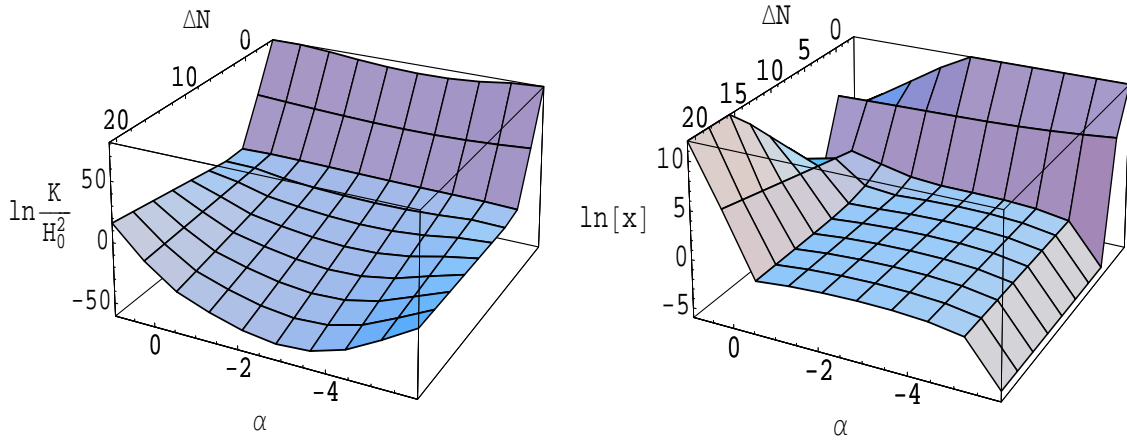

Figure 3: The kinetic term K/H_0^2 (left plot) and the dimensionless variable x (right plot), in logarithmic scales, as functions of α and ΔN, where $x \equiv (\gamma/2)(\dot{\sigma}/H)^2 = K/H^2 = (\gamma/2)\sigma'^2$. For $\alpha > 1$ (or $\alpha < -6$), $K < 0$ and hence $x < 0$, leading to a phantom type cosmology. For $-6 < \alpha < 0.45$, K/H_0^2 rapidly approaches zero. In all plots, where applicable, we have set $u_t = 10$.

One may express the field potential $V(\sigma)$ and the coupling constant $f(\sigma)$ as functions

of the field σ itself. As the second plot in figure 3 shows, for $\alpha \lesssim 0.45$, the parameter $x(N)$ is almost independent of N, implying that $N = \lambda_0\sigma +$ const; we will choose this last constant to be N_t. For $\Delta N \gg 0$, λ_0 ($\equiv \frac{1}{M_P}\sqrt{\frac{\gamma}{2x_0(\alpha)}}$) is a function of the slope parameter α only. This leads to the following expressions for the potentials:

$$V(\sigma) = V_0\, e^{-2\hat{\beta}\lambda_0\sigma}\, (\cosh(\beta\lambda_0\sigma))^2$$
$$\times \left[3 - \hat{\beta} + \beta\tanh(\beta\lambda_0\sigma) + 3\hat{u}_t\left(1 - \hat{\beta} + \beta\tanh(\beta\lambda_0\sigma)\right)e^{\alpha(\lambda_0\sigma)}\right], \quad (4.22)$$

$$f(\sigma) = \frac{V_0}{H_0^4}\,\hat{u}_t\, e^{(2\hat{\beta}+\alpha)\lambda_0\sigma}\,(\operatorname{sech}(\beta\lambda_0\sigma))^2\, e^{4\hat{\beta}N_t}, \quad (4.23)$$

where $V_0 \equiv M_P^2 H_0^2\, e^{-2\hat{\beta}N_t}$ and $\hat{u}_t \equiv u_t\, e^{\alpha N_t}$.
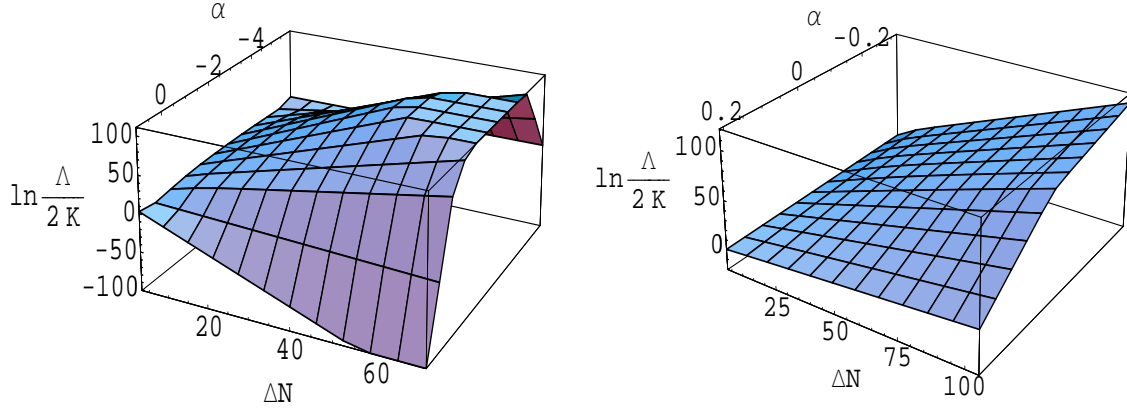

Figure 4: The ratio $\Lambda(\sigma)/2K(\sigma)$ in logarithmic scale, in different ranges for α.

For $-6 < \alpha < 1$, so $\hat{\beta} > \beta$, the Hubble parameter is a smoothly decreasing function of N in this regime. Typically, for $\alpha \simeq 1$ (or $\alpha \simeq -6$), since $\hat{\beta} \simeq \beta$, we have $H \simeq H_0$, $h \simeq 0$ and hence $w \simeq -1$: a stage where $\Lambda(\sigma)$ acts as a cosmological constant. While, for $1 > \alpha > -2$ (or $-6 < \alpha < -4$), we have $0 > h > -1$ and hence $-1 < w < -1/3$. When the value of α is decreased from unity, the kinetic term lowers towards zero at large N, while the parameter $x(N)$ is almost constant (cf figure 3). For $\alpha \lesssim 0.28$, the condition $\Lambda(\sigma) > 2K(\sigma)$ holds in general. But in our model this does not necessarily mean the existence of an accelerating phase; as the figure 4 shows, $\Lambda(\sigma) > 2K(\sigma)$ holds even if $-2 > \alpha > -4$; in this last case we get $h < -1$ and hence $w > -1/3$, implying a non-accelerating universe.

We will consider two epochs, an initial inflationary epoch where N is assumed to grow from an initial value $N_i \lesssim N_{early}$, and a late-time deceleration/acceleration phase where N becomes comparable to N_{late}, and $N \gtrsim N_{late}$ eventually holds. The universe starts to inflate when $N \gtrsim N_{early} + 0.5$. Consequently, at $N \gtrsim N_{early}$, we have a stage of inflation. For $\beta(N - N_{early}) \gtrsim 2.5$, the scalar field begins to freeze in, such that $w < -1/3$; the actual value of w depends on the value of slope parameter α, see [17]. After a certain number of

e-folds, $\Delta N = N_f - N_i$, our approximation, (4.16), with the *time* being *early* breaks down. For $N \lesssim N_{late}$, the universe is in a deceleration phase which implies that inflation must have stopped during the intermediate epoch. Sometime later, when $N > N_{late}$, subsequent evolution will be controlled by (4.16) with the *time* being *late*.

For $\beta \Delta N \gtrsim 2.5$, the variation in $h(N)$ is small, so we can approximate it by

$$h \simeq \begin{cases} -\hat{\beta} + \beta & (\Delta N \gg 0) \\ -\hat{\beta} - \beta & (\Delta N \ll 0). \end{cases} \tag{4.24}$$

Rearranging this approximation for h, we find $\alpha = \{c_1, c_2\}$, as defined in (4.12). This tells us that the earlier approximation, $h \approx h_0$, is a good 'early time' or 'late time' approximation *during* a given epoch. Also note that for $h \leq 0$ (and hence $w \geq -1$) we require $-6 \leq \alpha \leq 1$. Picking α outside this range leads to 'big rip' type cosmologies for which $h > 0$ (and hence $w < -1$) in some regions of field space.

4.3 Relaxation of dark energy

There might be a large shift in the Hubble expansion rate during inflation, viz $H_{before} \sim 10^{23} \, eV \gg H_{after} \sim 10^{-33} \, eV$. One could therefore ask whether the solutions we presented above explain a dynamical relaxation of vacuum energy (or scalar potential) to the present value of dark energy, $\sim 10^{-120} M_P^4$, after a significant period of inflation. This is quite plausible in our model. To quantify this, one considers the ratio of the Hubble parameters before and after N e-folds of inflation, which is given by

$$\varepsilon = \frac{\cosh \beta (\Delta N + N) \, e^{-\hat{\beta} N}}{\cosh \beta (\Delta N)}. \tag{4.25}$$

As an illustration consider that $\alpha \simeq 0.0143 \simeq 1/70$, so that $q_{ini} \simeq 0^-$ at $\Delta N \equiv N_{ini} - N_0 \simeq 0.33$. Assuming 70 e-folds of inflation, i.e. $N = 70$, we find

$$\varepsilon \simeq 1.36 \times 10^{-12}. \tag{4.26}$$

This value represents a shift in Hubble expansion rate, only during an accelerating epoch for which the initial value of the deceleration parameter is zero. Indeed, something like 70 e-folds of expansion, as usually considered, is the minimum for inflation, based on the assumption of a constant Hubble rate. Practically, one requires much more expansion than e^{70} between the Planck time and the present. As emphasized in [27], one might need $\tilde{N} \equiv \ln \frac{(aH)_f}{(aH)_i} \geq 70$, rather than $N \equiv \ln(a_f a_i) \geq 70$, to solve the various cosmological conundrums, including the flatness and horizons problems. The difference $N - \tilde{N}$, which is non-zero and positive, as long as $-6 < \alpha < 1$, would characterize the extra amount of expansion (e-folds) required by the decrease of H during inflation.

In our model, the total number of e-folds N required to get a small ratio, like $\epsilon \sim 10^{-56}$, depends on the value of the expansion parameter α, which may be related to the

slope of the potential coming purely from the Gauss-Bonnet coupling, $V_{GB}(\sigma) \equiv f(\sigma)\mathcal{G} = 3u(\sigma)H^2(1+h)$. This value would be minimum, $N \sim 125$, for $\alpha \gtrsim -2$ (or $\alpha \lesssim -4$), while it would be large, $\Delta N \gtrsim \mathcal{O}(300)$, for $\alpha \lesssim 1$ (or $\alpha \gtrsim -6$). In the former case, inflation could occur slowly since $\frac{d}{dt}(\frac{1}{aH}) \lesssim 0$, while, in the later case, it might occur more rapidly since $\frac{d}{dt}(\frac{1}{aH}) \ll 0$. For example, for $\alpha = 0$, the effective potential $\Lambda(\sigma) \sim 10^{-8}M_P^4$ decreases to the present value of dark energy, namely, $\Lambda_0 \sim 10^{-120}M_P^4$, when $N_{total} \sim 326$.

Assuming the universe has undergone a sufficient number of e-folds of expansion, like $N_{total} \gtrsim 125$, we find an effective scalar potential that can dynamically relax its value to the observed value of the cosmological constant, such that the field σ evolves towards its minimum and the equation of state parameter falls in the range $-1 < w < -1/3$.

4.4 Late-time acceleration

The deceleration parameter q, which may be parameterized as a function of red-shift factor z, is given by

$$q(z) = -\frac{1}{H^2}\frac{\ddot{a}}{a} = -(1+h).$$

The value of the function, h, for the current epoch must be determined by observation. For example, if the observed value of the deceleration parameter $q(z)$ corresponds to $\simeq -0.6$, then

$$q_{obs} = -1 - h_{obs}$$
$$\simeq -1 + \hat{\beta}_{late} - \beta_{late} \simeq -0.6\,, \qquad (4.27)$$

where we have made the late time approximation for h_{obs}. Solving $h_{obs} = -0.4$ we find

$$\alpha_{late} \simeq -0.0148 \quad \text{or} \quad \alpha_{late} \simeq -5.3851. \qquad (4.28)$$

Out of the two possible values of the slope parameter α, giving rise to the same value of deceleration parameter $q(z)$, we should prefer to take the smaller value of α, in terms of its absolute magnitude, so that the coupling $u(\sigma(N)) \sim f(\sigma)H^2$ varies only slowly with the expansion of the universe. In fact, the function $u(\sigma)$ is a measure of the strength of scalar coupling with the curvature terms: $u(\sigma)$ is likely to depend upon the gauge coupling strength. And, in the presence of matter sources, like standard model particles, the running of the gauge coupling could be small if $|\alpha| \lesssim \mathcal{O}(\frac{1}{N})$.

Given the definition of redshift

$$1 + z = a_{now}/a_z \qquad (4.29)$$

we can rewrite it as $1 + z = e^{|\Delta N|}$, allowing us to plot recent acceleration versus z. At late times, $N_0 = N_{late}$, we may observe a significant variation in h over the range of red-shift $z < 1.5$ (cf figure 5), in excellent agreement with observation [1].
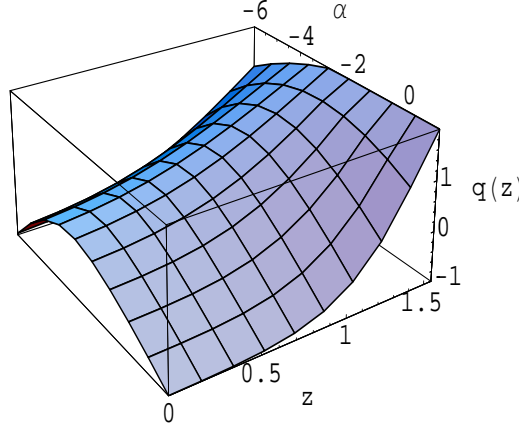
Figure 5: The solution is modelled such that $\ln(1+z) \equiv \Delta N = \ln \frac{a_f}{a_i}$. One can see a change from deceleration to acceleration as the red-shift factor z decreases in the range $0.5 \lesssim z \lesssim 1.5$.

4.5 Scalar potential: leading order behaviour

It might well be that, asymptotically (i.e. at late times), both the functions $h(N)$ and $u(\sigma(N))$ take (nearly) constant values. In this rather special limit

$$h(N) \to h_0 < 0, \quad u(\sigma(N)) \to u_0 > \frac{1}{h_0 - 1}, \tag{4.30}$$

we find the following solution

$$H = \frac{1}{c_0 - h_0 t}, \quad N = \int H \, dt, \tag{4.31}$$

$$\sigma\kappa = \pm\sqrt{\frac{2}{\gamma}}\sqrt{-h_0(1 + u_0 - u_0 h_0)}\, N + const, \tag{4.32}$$

$$V(\sigma) = \frac{H_0^2}{\kappa^2}\, e^{2h_0 N} \left(3 + h_0 - u_0 h_0(5 + h_0)\right) \equiv V_0\, e^{-\sigma/\sigma_0}, \tag{4.33}$$

where $\sigma_0\kappa = \sqrt{(u_0 - (1 + u_0)/h_0)/2\gamma}$. It is not difficult to see that, to leading order in σ, $f(\sigma) \propto e^{\sigma/\sigma_0}$. That is, the *ansätze* like $V(\sigma) = V_0\, e^{-\sigma/\sigma_0}$ and $f(\sigma) = f_0\, e^{\sigma/\sigma_0}$ characterize only the late time attractors, for which both $u(N)$ and $h(N)$ behave merely as constants.

Consider a more general case for which $h(N) \equiv \dot{H}/H^2 = h_0 < 0$, but $u(\sigma(N)) = u_0\, e^{\alpha N}$, $\alpha \neq 0$. In this case, σ is related to N via

$$\sqrt{\frac{\gamma}{2}}\,\sigma\kappa = \frac{\sqrt{c_1\, e^{\alpha N} - 4h_0}}{\alpha} - \frac{2\sqrt{-h_0}}{\alpha}\tanh^{-1}\left(\frac{\sqrt{c_1\, e^{\alpha N} - 4h_0}}{2\sqrt{-h_0}}\right), \tag{4.34}$$

where

$$c_1 \equiv 2u_0(\alpha - 2h_0)(1 - h_0 - \alpha). \tag{4.35}$$

– 15 –

Both signs of α may be allowed as long as $c_1 > 4h_0\,\mathrm{e}^{-\alpha N}$ holds. Indeed, $h(N) = H'/H \simeq 0$ is special case, for which

$$\sigma\kappa = \sqrt{\frac{2}{\gamma}}\,\sqrt{\frac{2u_0(1-\alpha)}{\alpha}}\,\mathrm{e}^{\alpha N/2} \tag{4.36}$$

and hence $\Lambda(\sigma) = 3M_P^2 H_0^2 \equiv \Lambda_0$. Note that, since the equation of state parameter $w = -1$, $\Lambda(\sigma)$ acts as a cosmological constant term in the regime $h \sim 0$. In this case σ can be real even if $\gamma < 0$, provided that both $\alpha < 0$ and $u_0 > 0$ hold simultaneously. Nevertheless, one should be more interested in a canonical scalar ($\gamma > 0$), in which case one now requires $0 < \alpha < 1$ and $u_0 > 0$. As we will see shortly, the slope parameter α in this range possesses an interesting feature that the spectrum of scalar (density) perturbation during inflation is almost flat, giving a scale invariant power spectrum, $n_s \simeq 1$.

5. Inflation and cosmological perturbations

It is generally believed that during inflation the inflaton and graviton field undergo quantum-mechanical fluctuations, leading to scalar (density) and tensor (gravity waves) fluctuations, which in turn would give rise to significant effects on the large-scale structure of the universe at the present epoch. The spectra of perturbations may provide a potentially powerful test of the inflationary hypothesis. One could ask whether the model presented here will add to the search for a conceivable physical basis for inflation. We believe this is possible.

5.1 Slow roll variables

In a standard scenario, one defines the slow roll variables, in terms of the field potential $V(\sigma)$:

$$\epsilon_v \equiv \frac{1}{2\kappa^2\sigma'^2}\left(\frac{V'}{V}\right)^2, \quad \eta_v \equiv \frac{1}{\kappa^2\sigma'^2}\left(\frac{V''}{V} - \frac{V'}{V}\frac{\sigma''}{\sigma'}\right). \tag{5.1}$$

As above, the prime denotes derivative with respect to N, not the field σ. These definitions for slow roll parameters may be justified since during the inflationary epoch $V(\sigma) \gg V_{GB}(\sigma)$ holds. Note that $V_{GB}(\sigma) = f(\sigma)\mathcal{G} = 3uH^2(1+h) \to 0$ as $h \to -1$. Because of the flatness of $V(\sigma)$, σ grows very slowly and essentially all the inflation occurs when $V(\sigma) \gg V_{GB}(\sigma)$. Slow-roll requires that $|\epsilon_v| \ll 1$, $|\eta_v| \ll 1$.

To impose the conditions on slow roll variables in a physically motivated and model-independent way, following ref. [27], we may define them in terms of the Hubble parameter $H(\sigma)$ and its derivatives:

$$\epsilon_H \equiv \frac{2}{\kappa^2}\left(\frac{H_\sigma}{H}\right)^2 = \frac{2}{\kappa^2\sigma'^2}\left(\frac{H'}{H}\right)^2, \tag{5.2}$$

$$\eta_H \equiv \frac{2}{\kappa^2}\frac{H_{\sigma\sigma}}{H} = \frac{2}{\kappa^2\sigma'^2}\left(\frac{H''}{H} - \frac{H'}{H}\frac{\sigma''}{\sigma'}\right), \tag{5.3}$$

where, as before, primes denote derivatives w.r.t. N. One also defines the following parameter, which is second order in slow-roll expansion:

$$\xi_H \equiv \frac{1}{2\kappa^2} \left(\frac{H_\sigma H_{\sigma\sigma\sigma}}{H^2} \right)^{1/2} = \left(\epsilon_H \eta_H - \sqrt{2\epsilon_H}\, \frac{\eta'_H}{\sigma'} \right)^{1/2}. \tag{5.4}$$

(The ξ_H defined above is not to be confused with the coupling function $\xi(\sigma)$ we defined before; here we are adopting the notations which are standard in literature and we will not refer to $\xi(\sigma)$ in this section). In fact, these definitions of slow roll variables may be of wider applicability than those defined in terms of $V(\sigma)$, as they are based on the fact that inflation occurs as long as $\frac{d}{dt}(\frac{1}{aH}) < 0$ holds. That is, during inflation, the comoving Hubble radius, $1/(aH)$, must necessarily decrease, so that physical scales can grow more rapidly than the Hubble radius. Inflation ends at $\frac{d}{dt}\left(\frac{1}{k}\right) = 0$, where $k \equiv aH \equiv a_e\, H(\sigma)\, e^{-\Delta N}$ is, by definition, a scale matching condition, where a_e is the value of the scale factor at the end of inflation, at which a coming mode crossed outside the (cosmological) horizon.

Let us analyze the last case considered above in some detail. First, note that, defining $\sigma \to \sigma/\sqrt{\gamma}$, we can always absorb the coupling constant γ into the slow-roll parameters. Without loss of generality, henceforth we define $\epsilon \equiv \epsilon_H/\gamma$, $\eta \equiv \eta_H/\gamma$, $\xi \equiv \xi_H/\gamma$; similar arguments would apply to other quantities, like, scalar and tensor indices, defined below.

At the level of approximation we are considering in this section, namely, $V(\sigma) \gg V_{GB}(\sigma)$, the tilt parameter, up to first order terms (ϵ and η), may be given by

$$n_s \simeq 1 - 4\epsilon + 2\eta. \tag{5.5}$$

(We will discuss below about the validity of such a relation in the presence of the Gauss-Bonnet term.) As is generally the case, a value $n_s < 1$ is easier to produce than $n_s > 1$ for the model we are considering here; $n_s \sim 1$ is the value that makes the (scalar) perturbation small in all scales, see, e.g. [28], for a review. In our model, $n_s > 1$ is possible only if one allows $\alpha > 1$ (or < -6). This last demand is however not a physically motivated one since the kinetic energy of the field σ is negative in at least some regions of field space.

In our model, the slow roll variables ϵ and η (and also ξ) can vary with both the exponent and coefficient of the coupling constant $u(\sigma(N))$, i.e. α and u_t, in addition to the number of e-folds, $N_f - N_i = \Delta N$ (cf figure 6). In turn, as the figures 7 and 8 show, it is possible to get a value of n_s close to unity in a wide range of u_t (or ΔN) by suitably choosing α. Interestingly, as shown in figure 9, a value of n_s in the range $[0.89, 1]$ may easily be obtained by taking different combinations of α and u_t. One obtains the value $n_s \sim 0.95$, in excellent agreement with recent observational data from WMAP, by taking

$$\Delta N \sim 70, \quad \alpha_{early} \sim -0.01, \quad u_{early} \sim 22. \tag{5.6}$$

For $\alpha < 0$, with a smaller value of ΔN, one also requires a smaller value of u_t; when $\Delta N = 50$ and $\alpha = -0.01$, we find $n_s \simeq 0.97$ for $u_{early} \simeq 31$, while $n_s \simeq 0.95$ for $u_{early} \simeq 18$.
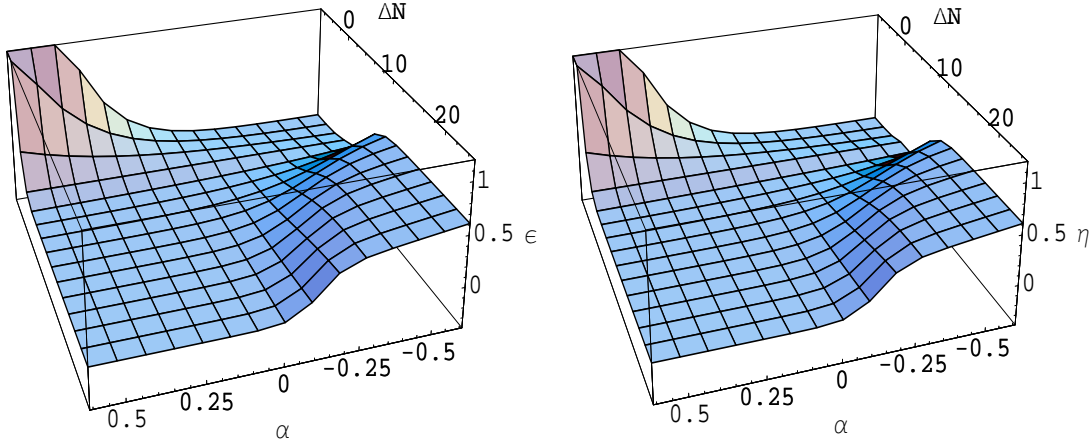
Figure 6: The slow roll variables ϵ and η as functions of α and ΔN. For a small and positive α, the universe is decelerating before inflation, $\Delta N < 0$, but ϵ (and also η) quickly takes a small value ($\epsilon, \eta \ll 1$) for $\Delta N \gtrsim 10$ and $\alpha > -0.2$, leading to inflation. In fact, $\epsilon \sim 0$ also for $\alpha \simeq -6$. Outside the range $-6 \leq \alpha \leq 1$, we get $\epsilon < 0$, leading to a phantom type cosmology. The condition for acceleration to occur is $\epsilon < 1$, so the universe is not accelerating for $-4 < \alpha < -2$.
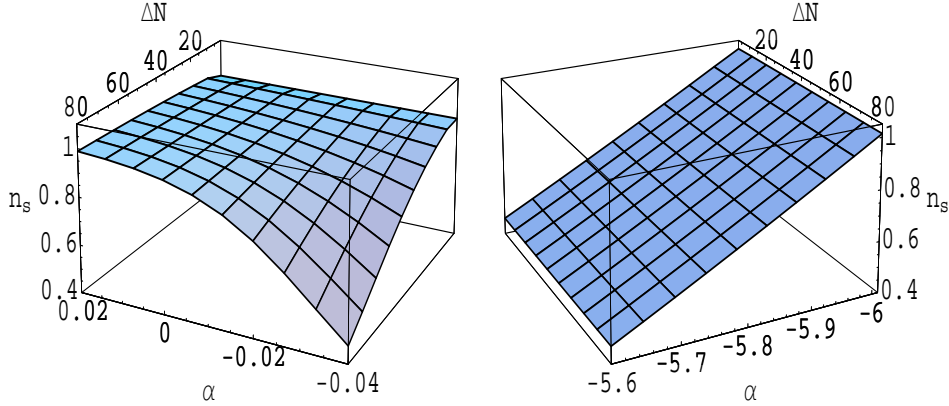

Figure 7: The spectral index n_s as a function of slope parameter α and ΔN. Both for $\alpha \gtrsim 0$ and $\alpha \ll 0$, the spectral index is independent of ΔN.

Of course, a positive value for the slope parameter, α, is also allowed, as long as $\alpha < 1$ holds. Note, for $\alpha > 0$ and $N > 0$ the function $u(\sigma(N)) \sim e^{\alpha N}$ grows with proper time, t, or logarithmic time N. But, in our construction, any contribution like this, coming from Gauss-Bonnet term is exactly cancelled with the *homogeneous* part (i.e. terms multiplied by $u(\sigma)$ and its derivatives) of the field potential $V(\sigma)$. Also, both the Hubble parameter H and $u(\sigma)H^2$ can be slowly decreasing functions of the number of e-folds, N, or the field σ.

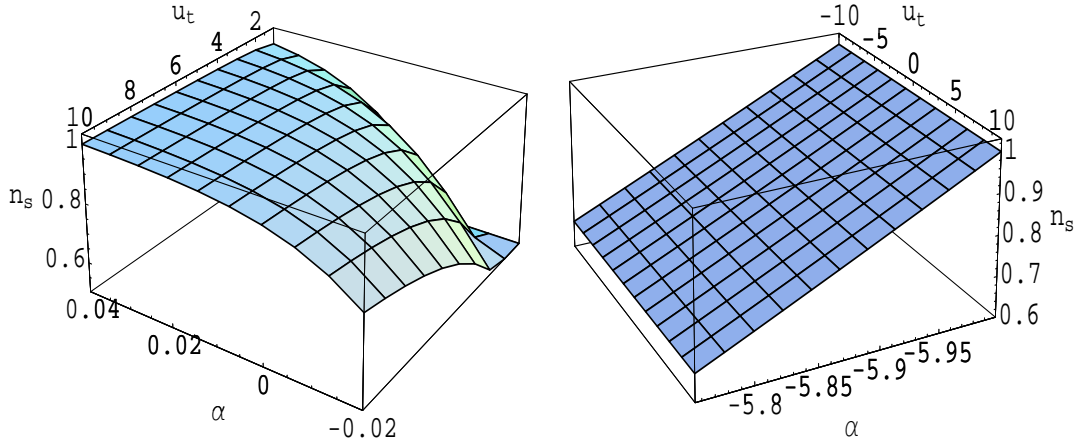Note that if $V(\sigma) = 0$, then Einstein gravity may not be an effective theory at low

Figure 8: The spectral index n_s as a function of α and u_t, with a fixed value of $\Delta N = 70$. The plots do not change much while including terms second order in slow-roll, viz, $n_s \simeq 1 - 4\epsilon + 2\eta - 2(1+c)\epsilon^2 - \frac{1}{2}(3-5c)\epsilon\eta + \frac{1}{2}(3-c)\xi^2$, where $c \simeq 5.08$, as defined in [27]. A small difference is that now a smaller value of u_t is required so as to get the same value of n_s.
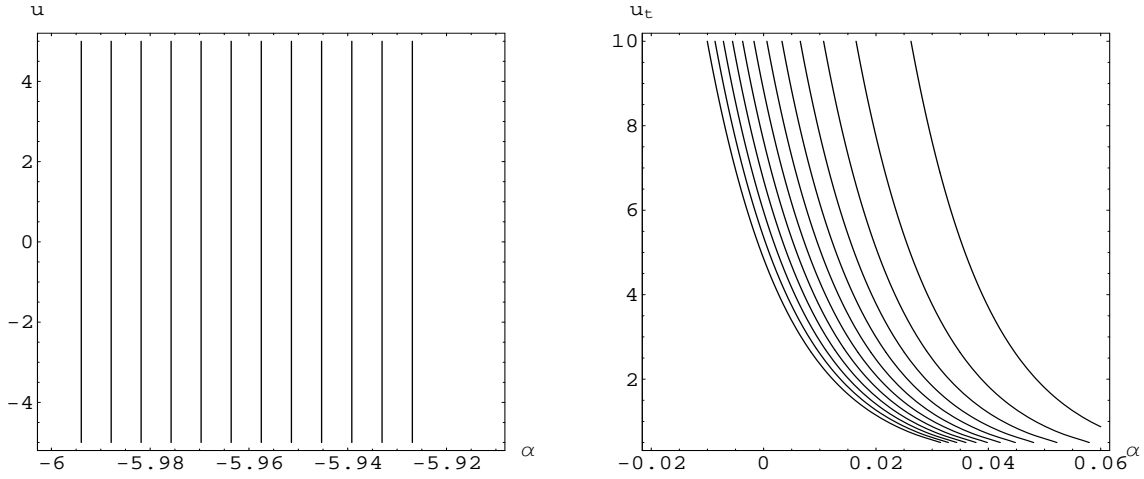

Figure 9: The contour plots of u_t vs α in the range $n_s = [0.89, 1]$, with a fixed value of $\Delta N = 70$. The spectral index, n_s, decreases (increases) from left to right in the left (right) plot; $n_s > 1$ can be obtained only for $\alpha < -6$ (or $\alpha > 1$). For $\alpha > 0$, more positive is the value of α, smaller will be the coefficient u_t giving rise to the same value of n_s. For $\alpha << 0$, however, n_s is insensitive to the value of u_t. For $\alpha > 0$, the signs of running of spectral index can be different between the $u_t < 0$ and $u_t > 0$ cases.

energy, since the term $u(\sigma)H^2$ can easily dominate the Einstein-Hilbert term, R/κ^2, which is proportional to $3H^2$. Even though the time scale for such effect to occur can be extremely large, given that α is very close to zero (i.e. $f(\sigma)$ scales nearly as $1/H(\sigma)^2$) and the coefficient u_t can be extremely small, this provides an additional justification for considering

an effective action with a non-trivial field potential, $V(\sigma)$.

5.2 Generation of perturbations

Note that for the solution (4.20) the scale factor of the universe evolves as

$$a(t) \sim (t + t_0)^{1/m} \sim |\tau|^{1/(m-1)}, \tag{5.7}$$

where $m \equiv \hat{\beta} - \beta < 1$ and τ is the conformal time. In the case of power-law inflation, such as this, the amplitudes for scalar and tensor fluctuations may be given by [29, 30]

$$A_s = \frac{4A_0^{(s)}}{5M_P^2} \left(1 + 0.46\epsilon - 0.73\eta\right) \frac{H\sigma'}{|h|}, \tag{5.8}$$

$$A_T = \frac{2A_0^{(T)} H}{5\sqrt{\pi} M_P} \left(1 - 0.27\epsilon\right). \tag{5.9}$$

Two remarks are in order. First, for each of these perturbations the value of H (and hence h) and σ' $(= \frac{d\sigma}{dN})$ must be evaluated when the wavelength of the perturbation becomes of the order H^{-1}. Practically, it is more convenient to specify them as functions of e-folds N before the end of inflation. Second, the above expressions, with $A_0^{(s)} = A_0^{(T)} = 1$, best approximate the results in a model with a self interaction potential $V(\sigma)$ alone [29], i.e. without the GB coupling. In fact, in the $f(\sigma) \neq 0$ case, A_0's are generally functions of $f(\sigma)$ and H, not unity. As long as the GB term is only sub-leading to the scalar potential $V(\sigma)$, the above results would be available to leading order.

Following [32, 33], one may calculate the coefficients related to scalar and tensor type perturbations

$$A_0^{(s)} = \frac{s_{(s)}^{-\nu/2}}{\sqrt{Q_{(s)}}}, \quad A_0^{(T)} = \frac{s_{(T)}^{-\nu/2}}{\sqrt{Q_{(T)}}}, \tag{5.10}$$

where $\nu \equiv \frac{3}{2} + \frac{m}{1-m}$ and

$$s_{(s)} \equiv 1 + \frac{4F^2}{1+2F} \left(2h + \frac{F - \ddot{f}}{1+2F}\right) \left(2x + \frac{6F^2}{1+2F}\right)^{-1}, \tag{5.11}$$

$$\sqrt{Q_{(s)}} \equiv \frac{1}{\kappa} \left|\frac{H}{\dot{\sigma}}\right| \left(2x + \frac{6F^2}{1+2F}\right)^{1/2} \left(\frac{1+2F}{1+3F}\right), \tag{5.12}$$

$$Q_{(T)} = 1 + 2F, \tag{5.13}$$

$$s_{(T)} Q_{(T)} = 1 + 2\ddot{f}, \tag{5.14}$$

where $F \equiv \kappa^2 \dot{f} H$ and $x \equiv \frac{\gamma}{2} \kappa^2 \left(\frac{\dot{\sigma}}{H}\right)^2$. In the limit $F \to 0$ (or $\dot{f} \to 0$), one recovers the standard result for which $s_{(s)} = s_{(T)} = 1$, $\sqrt{Q_{(s)}} = \sqrt{\gamma}$ and $\sqrt{Q_{(T)}} = 1$. As can be seen at the level of the field equations, equations (A.1)-(A.3), the condition $0 < |F| \ll 1$, or alternatively

$$F \equiv \kappa^2 \dot{f} H = \frac{H^2}{8} \left(\frac{u}{H^2}\right)' = \frac{1}{8}(u' - 2hu) \ll 1, \tag{5.15}$$

is equivalent to $V(\sigma) \gg V_{GB}(\sigma)$. This may be satisfied, for example, by choosing $u_t = 10$, $\Delta N = 70$ and $\alpha < 0$. $|F| < 1$ implies that $\dot{\sigma}H$ decays faster than the function $\frac{d\xi}{d\sigma}$ grows. Since $\ddot{f} = -\ddot{\xi} = -\dot{\sigma}^2 \frac{d^2\xi}{d\sigma^2} - \ddot{\sigma}\frac{d\xi}{d\sigma}$, calculating the above quantities requires knowledge about how the scalar field σ varies with time, which is model dependent.

Since $h \leq 0$, a small negative value of F would normally decrease the value of $s_{(s)}$ but would increase the value of $\sqrt{Q_{(s)}}$. This is opposite for the tensor modes: $s_{(T)}$ will increase but $\sqrt{Q_{(T)}}$ will decrease. Although A_s and A_T both vary with ΔN, or the choice of scale $k \equiv aH$ at which a comoving mode crossed outside the horizon, the ratio A_T/A_S is essentially independent of such a scale, or the number of e-folds, $\Delta N = N_f - N_i$. The tensor spectral index may therefore be given by [30, 31],

$$n_T \simeq -2\frac{A_T^2}{A_S^2}\left[2 - n_s - \frac{A_T^2}{A_S^2}\right]. \tag{5.16}$$

In the plots of figures 10 and 11, one may find a significant deviation for $F \sim -0.5$. Interestingly, for the solutions that we found above, this last last condition is extremely difficult to satisfy since it requires $u' - 2hu \simeq -4$, but $u' - 2hu \simeq 0_-$ at the end of inflation for $\alpha < 0$, and $u' - 2hu > 0$ for $\alpha > 0$.

Let us discuss about the validity of the relation, $n_s - 1 \simeq -4\epsilon + 2\eta$, in the presence of a non-trivial GB coupling. As shown in [34], in general terms, the spectral indices of the scalar and tensor-type perturbations may be given by

$$n_s - 1 = 3 - 2\nu_s, \quad n_T = 3 - 2\nu_t. \tag{5.17}$$

In our case, since $\nu_s = \frac{3}{2} + \frac{m}{1-m}$, it is sufficiently clear that $n_s - 1 = -\frac{2m}{1-m} \leq 0$. For the tree-level solutions found by Gasperini and Veneziano [35], the authors of [32] found the unpleasing result $n_s \simeq 4$ and $n_T \simeq 3$. The solutions given in [35] are characterized by the additional presence of a dilaton background $\phi(t)$, with $V(\phi) = 0$. This suggests, in our case, that the $V(\sigma) = 0$ case is not illuminating at least in view of inflationary paradigm. Indeed, the solutions found in [35], namely,

$$a(\tau) \propto |\tau|^{(1-\sqrt{3})/2}, \quad \phi(\tau) = -\sqrt{3}\ln|\tau| \tag{5.18}$$

are non-accelerating, so it is inconsistent in using non-inflationary solutions to estimate the inflationary parameters, like n_s.

In our model, it is quite possible that during an inflationary epoch the field σ changes only slowly, so the quantity $x(N) \sim (\dot{\sigma}/H)^2 = (\sigma')^2$ remains almost constant over exponentially large range of wavelengths, leading to an almost flat spectrum of perturbations of metric. Fluctuation of the field σ leads to the effect that the duration of inflation is increased due to a local delay of time near the exit from inflation, which may be given by

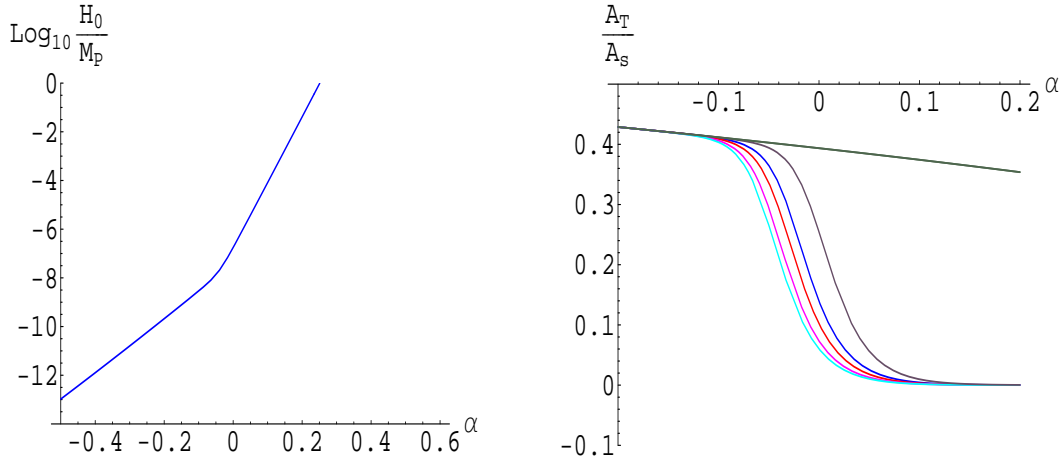$$\delta t = \frac{\delta\sigma}{\dot{\sigma}} \sim \frac{1}{2\pi\sigma'}. \tag{5.19}$$

Figure 10: (a) (left plot) The logarithm of the scale H_0/M_P vs α, giving rise to the value $A_S = 2 \times 10^{-5}$, which may be matched to the density contrast, δ_H, at Hubble-radius crossing. A particular value of α fixes the energy scale H_0 in terms of Planck mass, e.g., for $\alpha \simeq 0.0284$, $H_0 \simeq 10^{-6} M_P$. (b) (right plot) The tensor-to-scalar ratio, A_T/A_S, as a function of α: from left to right $u_t = 30, 20, 10, 5, 1$. This ratio is independent of the number of e-folds. The single (horizontal) line corresponds to $u_t = 0$, in which case A_T/A_S (and hence n_T) can be large ($r \lesssim 0.43$).
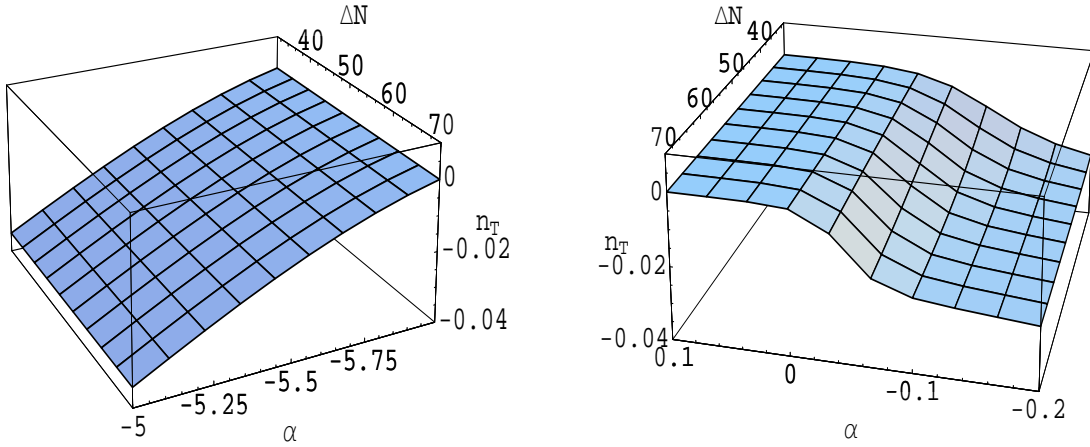

Figure 11: The tensor index A_T as a function of α and u_t, with fixed $\Delta N = 70$. The tensor modes are almost negligible, $n_T \simeq 0$, for $\alpha \gtrsim 0$ (or $\alpha \simeq -6$), or equivalently, for $\epsilon \ll 1$.

This may lead to a local density increase such that [11]

$$\delta_H \sim \frac{\delta\rho}{\rho} \sim \frac{H}{2\pi\,\sigma'}. \tag{5.20}$$

Inflation also leads to creation of perturbations of the field σ with wavelength greater than H^{-1}. The average amplitude of scalar perturbations during a typical time interval H^{-1} is

given by [36]

$$|\delta\sigma(x)| = \frac{H}{2\pi}. \tag{5.21}$$

Indeed, this result, in its more rigorous form $\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma^2\rangle = \frac{H^3}{4\pi^2}$, was independently obtained in [37]. A useful observational constraint is the following. If σ changed very slowly during inflation, then H/σ' remained almost constant over exponentially large range of wavelengths. The cosmic microwave background constraint is such that [38]

$$\delta_H \sim 1.92 \times 10^{-5}. \tag{5.22}$$

If this quantity is to be matched (precisely) with the scalar-index A_S, as the results given in [28] suggest, then, in our model, we can reproduce δ_H of this magnitude by taking, for instance, $H_0 \sim 10^{-7}M_P$ and $\alpha \sim -0.01$.

6. Towards reheating in an inflationary universe

It is essential to have a good reheating process after inflation in a cosmological model. Recall that in our model the time evolution of σ is given by (cf equation (2.10))

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\gamma}{2}\dot{\sigma}^2 + \Lambda(\sigma)\right) = -6H\left(\frac{\gamma}{2}\dot{\sigma}^2\right) - \delta. \tag{6.1}$$

where $\Lambda(\sigma) \equiv V(\sigma) - f(\sigma)\mathcal{G}$ and $\delta \equiv f(\sigma)\frac{\mathrm{d}\mathcal{G}}{\mathrm{d}t}$. Note that the effective potential for the dynamics of the scalar field is $\Lambda(\sigma)$ rather than $V(\sigma)$. The first term on the r.h.s. represents the energy loss caused by the expansion of the universe and the δ term represents the energy density per unit time which is drained from the field σ though time-variation of the Gauss-Bonnet term. Physically, such a term is expected since the modulus field σ, which obtains a non-zero mass due to its vacuum expectation value, $\langle\sigma\rangle$, is coupled to the curvature tensor.

Suppose that, initially, $\dot{\sigma} \sim 0$ and $\mathrm{d}\Lambda/\mathrm{d}t \sim 0$ ($\mathrm{d}\Lambda/\mathrm{d}\sigma$ is not essentially zero there). The evolution equation (6.1) then implies that $\delta \sim 0$ and hence

$$H \sim H_0\left(1 + \mathrm{e}^{4(N_0-N)}\right)^{1/4} \quad \Rightarrow \quad h \sim -\frac{\mathrm{e}^{4(N_0-N)}}{1 + \mathrm{e}^{4(N_0-N)}}, \tag{6.2}$$

where H_0 and N_0 are some integration constants. Suppose we start inflation at a point where $N < N_0$ [1] and $h \equiv \dot{H}/H^2 = H'/H \simeq -1$, so that the field equations satisfy

$$3\dot{\sigma}H \sim -\frac{1}{4}\left(\frac{\mathrm{d}f}{\mathrm{d}\sigma}\right)^{-1}, \quad \ddot{\sigma} \sim 12H\dot{\sigma}^3\frac{\mathrm{d}^2f}{\mathrm{d}\sigma^2} \ll 3H\dot{\sigma}. \tag{6.3}$$

For $N > N_0$, the universe experiences a nearly constant Hubble flow $H \sim H_0$, leading to an early inflation (exponential expansion) of the universe, i.e., $a \sim \mathrm{e}^{H_0 t}$. The physical Hubble radius H^{-1} increases with the number of e-folds N as the field σ rolls down its potential

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[1]One can define the scale factor as $a \equiv \mathrm{e}^{\omega(t)}$, so that $N \equiv \ln(a(t)) = \omega(t)$ and $N_0 = \omega_0$.

and gets out of its local extremum (a point of inflection or critical point). Inflation would be eternal (i.e. without an exit) if $\delta \simeq 0$ holds for all times. However, this is generally not the case since the Hubble flow is damped by an adequate cosmic friction term, and most of the evolution of the universe would be described by the $\delta \neq 0$ or $\frac{df(\sigma)}{d\sigma} \neq 0$ solution. At this point, one also notes that the term δ can change its sign between accelerating ($\frac{\ddot{a}}{a} > 0$) and decelerating ($\frac{\ddot{a}}{a} < 0$) solutions, since $\mathcal{G} = 24 \left(\frac{\dot{a}}{a} \right)^2 \frac{\ddot{a}}{a}$. That is, given that $f(\sigma)$ does not change its sign at the transition point, $\ddot{a} = 0$, the term δ acts as a (positive) friction term during an accelerating phase ($\frac{\ddot{a}}{a} > 0$), while it is opposite during a decelerating phase.

So far we have neglected the couplings of the scalar field to radiation (matter) fields. In the case the effective potential $\Lambda(\sigma)$ possesses a local minimum, such couplings may cause the rapid oscillatory phase to produce particles, leading to *reheating* [23, 24, 39] [2]. To this end, we can introduce some matter fields, which constitute ordinary matter and radiation field released by the decay of the field σ. For a homogeneous cosmological model with vanishing space curvature, we find

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{\kappa^2}{3} (\rho_\sigma + \rho_r), \tag{6.4}$$

$$\frac{d}{dt} \left(\frac{\gamma}{2} \dot{\sigma}^2 + \Lambda(\sigma) \right) = -6 \frac{\dot{a}}{a} \left(\frac{\gamma}{2} \dot{\sigma}^2 \right) - \delta, \tag{6.5}$$

where $\rho_\sigma = \frac{\gamma}{2} \dot{\sigma}^2 + V(\sigma) - 24 \dot{\sigma} H^3 \frac{df(\sigma)}{d\sigma}$. The δ term in (6.5) resembles a drag term, which transfers energy from the motion of σ and dumps it in the form of a radiation background. The equation for the evolution of the energy density in radiation (particle) is given by

$$\dot{\rho}_r + 4H \rho_r - \delta = 0. \tag{6.6}$$

Once reheating is completed the universe enters a standard radiation dominated FRW phase:

$$p_0 = \frac{1}{3} \rho_0 = \frac{1}{3} \rho_r, \quad \rho_0 \sim \frac{1}{a^4}, \quad H \sim \frac{1}{a^2} \sim \frac{1}{t}. \tag{6.7}$$

The field σ thereafter remains subdominant for most of the time and only at late times, $N \gtrsim N_{late}$, when the potential becomes sufficiently shallow, does one get acceleration, $w < -1/3$. This would then make the model complete.

A few remarks are in order. The first is related to a reheating process after inflation, the details of which depend upon a choice of the potential $V(\sigma)$ and the coupling $f(\sigma)$. More generally, it is essential to know whether or not the effective potential $\Lambda(\sigma)$ has a minimum. So far in this section we have not specified the form of $\Lambda(\sigma)$; our discussion of particle production above is rather qualitative.

Note that for the solution given in subsection (4.2), $\Lambda(\sigma)$ does not have a local minimum, rather it has a local maximum (cf figure 12). In this sense the conventional reheating

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[2]However, in non-oscillatory models, for example a model of quintessence characterized by a single exponential potential, the instant preheating mechanism proposed by Felder et al [40] might be more efficient for particle production, which works even for a potential without a minimum.
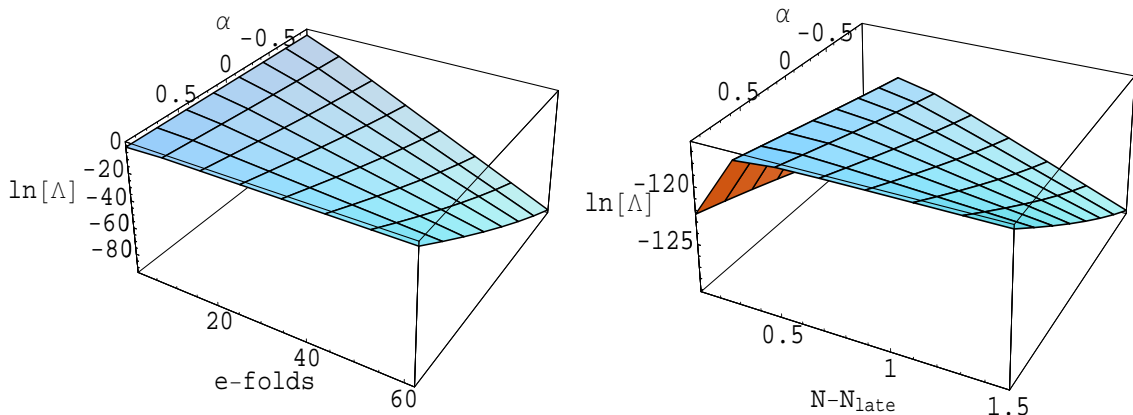
Figure 12: Approximate behaviours of the effective potential $\Lambda(\sigma)$ (not to scale) at early (left plot) and late (right plot) times; we have chosen the free (integration) parameters $H_{0,\text{early}}$ and $H_{0,\text{late}}$ such that the plots mimic a realistic feature that $\Lambda(\sigma)_{\text{early}} \gg \Lambda(\sigma)_{\text{late}}$.

process may not work and an alternative reheating method need to be employed: an efficient method of reheating is the *instant preheating* proposed by Felder et al [40]. In this case the field σ decays when it rolls down the potential, thereby producing heavy particles. However, details of the reheating process may be somewhat different and complicated in our model for at least two reasons. Firstly, as compared to a standard model, there is an extra friction-like term, namely δ, which is non-zero as long as $\frac{\mathrm{d}f(\sigma)}{\mathrm{d}\sigma} \neq 0$. Secondly, in our ansatz for $u(\sigma)$ ($\equiv 8\kappa^2 f(\sigma)H^2$), we assumed that $u(\sigma) \propto \mathrm{e}^{\alpha_t N}$, where the time t can be *early* or *late*. In order to retrieve the full potential, we may have to sew these two potentials in some way, possibly creating a barrier (and hence a minimum) somewhere between them. Work in this direction is currently underway. Numerical studies show that for an ansatz of the type $u(\sigma) = u_{\text{early}}\, \mathrm{e}^{\alpha_t N} + u_0(N) + u_{\text{late}}\, \mathrm{e}^{\alpha_{late} N}$, where $u_0(N) \sim \mathcal{O}(1)$, $\Lambda(\sigma)$ can have an effective local minimum, where inflation could end. This is in accordance with our observation in subsection (4.2), that inflation must have stopped during the intermediate phase. In fact, some of our solutions presented in the Appendix [3] possess an effective local minimum. In any case, the *instant preheating* proposed in [40] is perhaps the most efficient method of particle production in our model, as it works even for a potential without a minimum.

The second issue is related to the nucleosynthesis bound. As is known, for inflation driven by a scalar field, there exists a tight constraint on the allowed magnitude of $\Omega_\sigma \equiv \frac{\rho_\sigma}{\rho_\sigma + \rho_m} < 0.1 - 0.2$ at the time of nucleosyntheis; see, e.g., [41]. This often places a constraint on the model parameters, for instance on the slope of the (effective) potential. One could

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[3]In the Appendix we do not make any specific ansatz for $u(\sigma)$ but instead allow some of the variables to take their limiting values, like $x \simeq \text{const}$ and $h \simeq \text{const}$

ask whether or not the nucleosynthesis bound will not be violated in our model. To address this question properly, we would need to know the precise form of the (effective) potential.

It is always possible to constrain some of the parameters in our model by allowing $V(\sigma)$ and $f(\sigma)$ to take some specific (functional) form. In subsection (4.2), we found a solution by making a specific ansatz for $f(\sigma)$ (or $f(\sigma)H^2$). For this solution, when the gravity is coupled to matter and radiation, the necleosynthesis bound, in terms of the allowed magnitude of Ω_σ, constrains the values of the parameters β and $\hat{\beta}$, namely $4|\hat{\beta} - \beta|^2\lambda_0^2 > \frac{n}{\Omega_\sigma^{\max}}$, where $n = 3$ (4) for matter (radiation) and $\lambda_0 \equiv \frac{1}{M_P}\sqrt{\frac{\gamma}{2x_0(\alpha)}} = |\sigma'|_{\beta\Delta N > 2}$. The nucleosysthesis bound may not be violated for $(\hat{\beta} - \beta)\lambda_0 \gtrsim \sqrt{5}$, given that $\Omega_\sigma^{\max} \lesssim 0.2$; in arriving at this result we have assumed that the effective potential after a certain number of e-folds $(\beta\Delta N > 2)$ is approximated by $\Lambda(\sigma) \propto e^{-2(\hat{\beta}-\beta)\lambda_0\sigma}$, since $\tanh(\beta\Delta N) \to 1$. Also note that in this limit, $V(\sigma) \propto e^{-2(\hat{\beta}-\beta)\lambda_0\sigma}$, but with a different proportionality constant.

An inflationary type potential that we arrived at from studying symmetries of the field equations is approximated by $V(\sigma) = V_0 \exp[-p(\sigma)\kappa\sigma]$, with p being a function of the field σ or the number of e-folds N. Thus, the reheating process discussed in [42] may be useful in our model. Finally, we would like to note that our model might inherit some of the features of *quintessential inflation* proposed by Peebles and Vilenkin [43] in which the potential consists of two parts: $\lambda(\sigma^4 + M^4)$ ($\sigma < 0$) for inflation and $\lambda M^8/(\sigma^4 + M^4)$ ($\sigma \geq 0$) for quintessence; both models can lead to tracker solutions at late times, though the forms of scalar potentials are quite different.

7. Conclusions

In this paper we presented an analysis of accelerating/inflationary cosmologies by introducing in the effective action a field dependent Gauss-Bonnet coupling, other than a standard field potential for the field σ. We find that the dark energy hypothesis fits into a low energy gravitational action where a scalar field is coupled to the curvature squared terms in Gauss-Bonnet combination. It is established that a GB scalar-coupling can play an important and interesting role in explaining both the early and late-time evolutions of the universe as well as providing a mechanism for reheating.

That we are able to explain accelerating universes using exact cosmological solutions in a modified Gauss-Bonnet theory, leading to a small deviation from the $w = -1$ prediction of non-evolving dark energy (or a cosmological constant) is likely to have a serious impact in search of a viable dark energy model. Our work also provides extension of quintessence (or time-varying Λ) model in which part of the dark energy comes from a field dependent Gauss-Bonnet interaction term.

One of the key results is this: in the absence of a Gauss-Bonnet coupling, the tensor/scalar ratio is usually non-zero. However, with a non-trivial scalar Gauss-Bonnet coupling, i.e., $f(\sigma) \neq$, or effectively, $u(\sigma) \equiv f(\sigma)H^2 \sim e^{\alpha N} \neq 0$, such a ratio can be negligibly

small if the expansion parameter α takes a small positive value, $\alpha \gtrsim 0.1$, and hence $n_s \lesssim 1$ and $n_T \simeq 0$, leading to Harrison-Zel'dovich spectrum. Unlike a naive expectation, the inclusion of a (scalar) field dependent Gauss-Bonnet coupling $f(\sigma)$, in addition to a field potential $V(\sigma)$, into the effective action, could make the observability of tensor/scalar ratio and related inflationary parameters more achievable.

We emphasize that, in contrast to previous analysis, our calculations have all been implemented by the functional forms of the scalar potential and GB scalar coupling, as suggested by the symmetry of the field equations, rather than choosing particular model dependant forms for them. We have given in the Appendix the exact solutions for some special cases, about which a general comparison can be made in terms of the homogeneous solutions we presented in the bulk part of the paper.

Regardless of whether the model studied here appears natural or otherwise, it should be observation that determines whether or not it is correct. The current and future observations might make stronger demand on theoretical precision of inflationary parameters, including, the scalar and tensor spectral indices, and are certain to constrain a number of parameters of our model tightly, including the Gauss-Bonnet coupling constants.

Acknowledgements

8. Appendix

In terms of the dimensionless variables defined in equation (4.1), the equations of motion, (2.8)-(2.10), form a set of second order differential equations:

$$0 = -3 + x + y - 3(u' - 2hu), \tag{A.1}$$

$$0 = u'' - 2h'u - hu' + 2u' - 4hu - 2h^2u + x - y + 2h + 3, \tag{A.2}$$

$$0 = x' + 2(h + 3)x + y' + 2hy - 3(h + 1)(u' - 2hu). \tag{A.3}$$

where

$$X' \equiv \frac{dX}{dN} = a\frac{dX}{da} = \frac{1}{H}\frac{dX}{dt}, \tag{A.4}$$

so that

$$N = \int H\,dt = \ln\left(\frac{a(t)}{a_0}\right) \tag{A.5}$$

measures the logarithmic expansion of the universe.

Note, the logarithmic time, or the number of e-folds, N, is a monotonically increasing function of proper time t. However, its sign depends on the assumption of what the scale

a_0 represents. If a_0 is the initial value of $a(t)$, such that $a(t) \geq a_0$, then N starts from zero and take a large positive value when $a(t) \gg a_0$. However, if one wants a_0 to represent the present value of the scalar factor, then N usually starts from a large negative number when $a(t) \ll a_0$ and becomes positive only when $a(t) > a_0$.

Here we would like to present some exact solutions for some special cases, about which a general expression can be obtained for various parameters of the model, like, the field potential $V(\sigma)$ and the coupling constant $f(\sigma)$, in terms of the scalar field σ.

8.1 $x(N) = x_0$ and $h(N) = h_0$

As a reasonable approximation, at late times, consider that the kinetic term $K(\sigma) \propto H^2(\sigma)$, so $\dot{\sigma}/H \simeq$ const. Specifically, when $x \equiv \kappa^2 \frac{K(\sigma)}{H^2(\sigma)} = x_0$, the field equations reduce to

$$0 = -3 + x_0 + y - 3(u' - 2hu),$$
$$0 = u'' - (h+1)u' + 2h(1 + u - hz) - 2uh' + 2x_0. \tag{A.6}$$

$h \equiv \dot{H}/H^2$ is a kind of slow-roll variable, and thus may be treated as a constant in at least some region of field space. This is the case, for example, for power-law inflation, namely, $a(t) \propto t^{1/m}$ where $m < 1$. For $x = x_0$ and $h = h_0$, we find the following interesting solution:

$$u(N) = u_0 + u_1 \, e^{2h_0 N} + u_2 \, e^{(1-h_0)N}, \tag{A.7}$$
$$y(N) = y_0 + y_1 \, e^{(1-h_0)N}, \tag{A.8}$$

where $y_0 = 3 + h_0(1 - 5u_0 - h_0 u_0)$, $y_1 \equiv 3u_2(1 - 3h_0)$, and u_0, u_1, u_2 are the integration constants. We also find that

$$\kappa\sigma = \pm N \sqrt{\frac{2x_0}{\gamma}} + const, \quad x_0 \equiv -h_0(1 + u_0 - h_0 u_0). \tag{A.9}$$

This further implies that

$$V(\sigma) = \frac{H_0^2}{\kappa^2} \left(y_0 \, e^{2h_0 N} + y_1 \, e^{(1+h_0)N} \right), \tag{A.10}$$
$$V_{GB}(\sigma) = \frac{3H_0^2}{\kappa^2}(1 + h_0) \left(u_0 \, e^{2h_0 N} + u_1 \, e^{4h_0 N} + u_2 \, e^{(1+h_0)N} \right). \tag{A.11}$$

Using (A.9), one may express these potentials as functions of the field σ itself, namely

$$V(\sigma) \sim V_0 \, e^{-\sigma/\sigma_0} + V_1 \, e^{\frac{h_0+1}{2h_0}(\sigma/\sigma_0)}, \tag{A.12}$$
$$f(\sigma) \sim f_0 \, e^{\sigma/\sigma_0} + f_1 + f_2 \, e^{\frac{3h_0-1}{2h_0}(\sigma/\sigma_0)}, \tag{A.13}$$

where $\mp\kappa\sigma_0 h_0 \equiv \sqrt{\frac{x_0}{2\gamma}} > 0$. By combining the above expressions, we find

$$\Lambda(\sigma) = \frac{H_0^2}{\kappa^2} \left(A \, e^{2h_0 N} + B \, e^{4h_0 N} + C \, e^{(1+h_0)N} \right), \tag{A.14}$$

where $A \equiv 3(1-u_0)+h_0(1-8u_0-u_0h_0)$, $B \equiv -3(1+h_0)u_1$ and $C \equiv -12h_0u_2$. Equivalently,

$$\Lambda(\sigma) \equiv \Lambda_0 \, e^{-\sigma/\sigma_0} + \Lambda_1 \, e^{-2\sigma/\sigma_0} + \Lambda_2 \, e^{-\frac{1+h_0}{2h_0}(\sigma/\sigma_0)}. \tag{A.15}$$

Thus, since $h_0 \leq 0$, it is the term proportional to e^{4h_0N} or $e^{(1+h_0)N}$ which is of greater interest at early ($N \lesssim 0$) or late ($N \gg 0$) times. In particular, when $h_0 \sim 0$, we find

$$V(\sigma) = 3M_P^2 H_0^2(1 + u_2 \, e^N), \quad V_{GB}(\sigma) = 3M_P^2 H_0^2(u_0 + u_1 + u_2 \, e^N). \tag{A.16}$$

In this case, since $\Lambda(\sigma) \sim 3M_P^2 H_0^2(1 - u_0 - u_1)$, $\Lambda(\sigma)$ acts as a cosmological constant term, which is positive for $u_0 + u_1 < 1$.

8.2 $u(N) = u_0$ and $x(N) = x_0$

Let us recall that in the original action

$$\frac{R}{2\kappa^2} = \frac{3H^2}{\kappa^2}(2 + h), \tag{A.17}$$

$$f(\sigma)\mathcal{G} = 24f(\sigma)H^4(1 + h) = \frac{3H^2}{\kappa^2}(1 + h)\,u(\sigma). \tag{A.18}$$

In a general situation, the coupling constant $f(\sigma)$ increases with logarithmic time, N, while the Hubble expansion rate, $H(N)$, decreases with N. To this end, let us consider the case where the coupling constant $f(\sigma)$ scales as $1/H^2$, so that the function $u(\sigma)$ is constant, i.e. $u(\sigma) \simeq \text{const} \equiv u_0$. In general, we would require $u_0 < \frac{2+h}{1+h}$, because any contribution coming from the higher order curvature corrections in low energy string effective actions should be, at least in a low energy scale, an order of magnitude smaller than the contribution from the Einstein-Hilbert term. Also, the function $u(\sigma(N))$, which is likely to depend upon the gauge coupling strength, has to be an extremely slow varying function of proper time so that it assumes a nearly constant value.

Let us further demand that the kinetic term $K(\sigma)$ scales as H^2, so that $x(N) = \kappa^2 \frac{K}{H^2} \equiv x_0$. In this case, the explicit solution is given by

$$y(N) = 3 - x_0 - 6u_0h(N), \quad h(N) = h_1 + \beta \tanh \beta(N - N_0), \tag{A.19}$$

where

$$h_1 \equiv \frac{u_0 + 1}{2u_0}, \quad \beta \equiv \frac{\sqrt{(u_0 + 1)^2 + 4x_0u_0}}{2u_0}. \tag{A.20}$$

Further, a simple calculation shows that

$$\sigma = \pm\frac{\sqrt{2}}{\kappa}\sqrt{\frac{x_0}{\gamma}}N + const, \tag{A.21}$$

$$H = H_0 e^{h_1 N} \cosh \beta(N - N_0), \tag{A.22}$$

where N_0 is arbitrary. For a canonical scalar, $\gamma > 0$, we have $x_0 \geq 0$. A smaller value of x_0 makes the potential flatter and hence increases the period of inflation.

Using (A.9), or equivalently $N \equiv \sqrt{\frac{\gamma}{2x_0}}\,\sigma\kappa +$ const (we will choose this last constant to be N_0), we find

$$V(\sigma) = H_0^2 M_P^2\, e^{\frac{u_0+1}{u_0}\sqrt{\frac{\gamma}{2x_0}}\sigma\kappa}\cosh^2\beta\left(\sqrt{\frac{\gamma}{2x_0}}\sigma\kappa\right)$$

$$\times\left[-x_0 - 3u_0 - 6u_0\beta\tanh\beta\left(\sqrt{\frac{\gamma}{2x_0}}\sigma\kappa\right)\right], \tag{A.23}$$

$$f(\sigma) = \frac{u_0 M_P^2}{8H_0^2}\, e^{-\frac{u_0+1}{u_0}\sqrt{\frac{\gamma}{2x_0}}\sigma\kappa}\,\mathrm{sech}^2\beta\left(\sqrt{\frac{\gamma}{2x_0}}\sigma\kappa\right). \tag{A.24}$$

Here a natural choice of u_0 is $-1 < u_0 < 0$, so that the Hubble parameter $H(\sigma)$ is a monotonically decreasing function of the logarithmic time $\ln(a)$, or N. The effective potential is given by

$$\kappa^2\Lambda(\sigma) = H_0^2 M_P^2\, e^{\frac{u_0+1}{u_0}\sqrt{\frac{\gamma}{2x_0}}\sigma\kappa}\cosh^2\beta\left(\sqrt{\frac{\gamma}{2x_0}}\sigma\kappa\right)$$

$$\times\left(3 - x_0 - 3u_0 - 9u_0\left[h_1 + \beta\tanh\beta\left(\sqrt{\frac{\gamma}{2x_0}}\sigma\kappa\right)\right]\right). \tag{A.25}$$

In theories of quintessence, one Taylor expands the potential $\Lambda(\sigma)$ about the minimum, so as to obtain a dynamical scale for the mass of the quintessence field. This potential therefore merits further study.

8.3 $u(N) = u_0$ and $h(N) = h_0$

Consider, again, the case where $u = u_0$, but now instead of $x(N) \simeq$ const, we demand that $h(N) = \dot{H}/H^2 \simeq$ const. The solutions for $x(N)$ and $y(N)$ are given by

$$x = -h_0(1 + u_0 - h_0 u_0), \quad y = 3 + h_0(1 - 5u_0 - h_0 u_0). \tag{A.26}$$

One also finds the following relationships:

$$\sigma = \pm\frac{\sqrt{2}}{\kappa}\sqrt{\frac{x}{\gamma}}N + const, \quad H = H_0\, e^{h_0 N}. \tag{A.27}$$

The effective potential is then given by

$$\Lambda(\sigma) = V_0 e^{2h_0 N}\left(3 + h_0 - 3u_0 - 8h_0 u_0 - h_0^2 u_0\right) \sim \Lambda_0\, e^{-\sigma/\sigma_0}. \tag{A.28}$$

8.4 $x(N) = x_0$ and $y(N) = y_0$

Finally, as one more special case, let us consider that both the potential term $V(\sigma)$ and the kinetic term $K(\sigma) = \frac{\gamma}{2}\dot\sigma^2$ scale with H^2; presumably, with different proportionality constants. For $y = \kappa^2\frac{V(\sigma)}{H^2} = y_0$ and $x = \kappa^2\frac{K}{H^2} = x_0$, the solution is given by

$$h(N) = -\frac{3 + 5x_0 - y_0}{x_0 + y_0 + 3} \equiv h_0, \quad u(\sigma(N)) = \frac{(3 - x_0 - y_0)h_0}{6} - u_1\, e^{2h_0 N}, \tag{A.29}$$

where u_1 is an integration constant. The effective potential is therefore

$$\Lambda(\sigma) = V_0 \, \mathrm{e}^{2h_0 N} \left[3u_1(1 + h_0)\mathrm{e}^{2h_0 N} + 2\left(y_0 + \frac{x_0(x_0 - 2y_0 - 3)}{5x_0 - y_0 + 3} \right) \right]$$
$$\simeq \Lambda_0 \, \mathrm{e}^{-\sigma/\sigma_0} + \Lambda_1 \, \mathrm{e}^{-2\sigma/\sigma_0}. \tag{A.30}$$

From the above results, it is clear that at late times, $\sigma \gg \sigma_0$, the leading order contribution to the potential comes from the term proportional to $\mathrm{e}^{-\sigma/\sigma_0}$.

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